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COMMENT

Integrals involving Airy functions

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Abstract. We show how to evaluate a large number of integrals involving Airy functions. The method uses the fact that the Wronskian has a very simple form.

A recent paper by Wille and Vennik (1985) presented a derivation of the result

$$\int_0^\infty \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x = \pi/8 \tag{1}$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the standard Airy functions. They also present a result for the indefinite integral of this same integrand. The purposes of this comment are to provide an alternative derivation of the results of Wille and Vennik, to point out that their indefinite integral has an incorrect sign and to show that our alternative derivation leads to a generalisation of their result, enabling us to derive a large number of additional integrals.

We follow the practice of Wille and Vennik of using (ASn) to refer to a formula from the compilation of Abramowitz and Stegun (1965). Their method of deriving (1) uses (AS 10.4.69) to introduce the modulus and phase of the two kinds of Airy functions. It appears to be more direct to work first with the indefinite integral. One takes the derivative of

$$-\frac{\pi}{2}\frac{\mathrm{Ai}^2(x)}{\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x)} \tag{2}$$

and uses the fact that the Wronskian is given by (AS 10.4.10)

$$\mathrm{Ai}(x)\mathrm{Bi}'(x)-\mathrm{Ai}'(x)\mathrm{Bi}(x)=1/\pi. \tag{3}$$

The result is

$$\int \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x = -\frac{\pi}{2}\frac{\mathrm{Ai}^2(x)}{\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x)}. \tag{4}$$

This answer is the same as obtained by Wille and Vennik except for the sign. The definite integral (1) can now be obtained easily by using the fundamental theorem of integral calculus, since $\mathrm{Ai}(0)=\mathrm{Bi}(0)/\sqrt{3}$ (AS 10.4.4).

During the process of taking the derivative, one is struck by the symmetry of the integrand and the simplicity that results from the fact that the Wronskian does not depend on x. By the method of direct differentiation it is easy to show that an alternative result is

$$\int \frac{\mathrm{Ai}(x)\mathrm{Bi}(x)}{(\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x))^2}\,\mathrm{d}x = \frac{\pi}{2}\frac{\mathrm{Bi}^2(x)}{\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x)}. \tag{5}$$

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In similar fashion, the simplicity of the Wronskian can be used to show that the integral in (4) is not the most general result that could have been obtained. For example,

$$\int \frac{\mathrm{Ai}^{n-1}(x)\mathrm{Bi}^{n-1}(x)}{(\mathrm{Ai}^n(x)+\mathrm{Bi}^n(x))^2}\,\mathrm{d}x = \frac{\pi}{n}\,\frac{\mathrm{Bi}^n(x)}{\mathrm{Ai}^n(x)+\mathrm{Bi}^n(x)} \tag{6}$$

is a straightforward generalisation that can be checked easily by differentiation. In the special case where $n=2$, this result is the same as (5). If $n=1$,

$$\int \frac{\mathrm{d}x}{\mathrm{Ai}^2(x)+2\mathrm{Ai}(x)\mathrm{Bi}(x)+\mathrm{Bi}^2(x)} = \frac{\pi\mathrm{Bi}(x)}{\mathrm{Ai}(x)+\mathrm{Bi}(x)}. \tag{7}$$

Success at finding these results encouraged us to cast about for other examples. We were not disappointed, as can be seen from the following list:

$$\int \frac{\mathrm{d}x}{\mathrm{Ai}^2(x)} = \pi\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)} \tag{8}$$

$$\int \frac{\mathrm{d}x}{\mathrm{Bi}^2(x)} = -\pi\frac{\mathrm{Ai}(x)}{\mathrm{Bi}(x)} \tag{9}$$

$$\int \frac{\mathrm{d}x}{\mathrm{Ai}(x)\mathrm{Bi}(x)} = \pi\ln\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)} \tag{10}$$

$$\int \frac{\mathrm{Bi}^n(x)\,\mathrm{d}x}{\mathrm{Ai}^{n+2}(x)} = \frac{\pi}{n+1}\left(\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)}\right)^{n+1}. \tag{11}$$

Such simple results caused us to suspect that there may be a more general formula; this hope was also rewarded. Let f and F be any two functions such that

$$f = F'. \tag{12}$$

Then

$$\int \frac{1}{\mathrm{Ai}^2(x)} f\left(\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)}\right)\mathrm{d}x = \pi F\left(\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)}\right) \tag{13}$$

and

$$\int \frac{1}{\mathrm{Bi}^2(x)} f\left(\frac{\mathrm{Ai}(x)}{\mathrm{Bi}(x)}\right)\mathrm{d}x = -\pi F\left(\frac{\mathrm{Ai}(x)}{\mathrm{Bi}(x)}\right). \tag{14}$$

These two results are also very easily checked by differentiation. They are general enough so that all the integrals earlier in this comment are special cases of them. Clearly the list could be extended to whatever limit is set by one's patience. For instance,

$$\int \frac{\mathrm{d}x}{\mathrm{Ai}^2(x)+\mathrm{Bi}^2(x)} = \pi\tan^{-1}\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)} \tag{15}$$

$$\int \frac{\mathrm{Bi}^n(x)}{\mathrm{Ai}^{n+2}(x)}\exp\left(\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)}\right)^{n+1}\mathrm{d}x = \frac{\pi}{n+1}\exp\left(\frac{\mathrm{Bi}(x)}{\mathrm{Ai}(x)}\right)^{n+1}. \tag{16}$$

These results work so well with Airy functions because of the simplicity of the Wronskian. Other functions could be used instead. If $y_1(x)$ and $y_2(x)$ have as their Wronskian

$$w(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{17}$$

then

$$\int \frac{w(x)}{y_1^2(x)} f\left(\frac{y_2}{y_1}\right) \mathrm{d}x = F\left(\frac{y_2}{y_1}\right). \tag{18}$$

For the case of ordinary Bessel functions

$$J_n(x) N_n'(x) - J_n'(x) N_n(x) = 2/(\pi x) \tag{19}$$

is the Wronskian (AS 9.1.16). Therefore

$$\int \frac{1}{x J_n^2(x)} f\left(\frac{N_n(x)}{J_n(x)}\right) \mathrm{d}x = \frac{\pi F}{2}\left(\frac{N_n(x)}{J_n(x)}\right). \tag{20}$$

References

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